



OPTIMAL CODE

LENGTHS

In order to approach
lower bound $H(X)$

Optimal set of lengths:
 $l_0, l_1, \dots, l_{n-1}$

✓ satisfy Kraft Inequality

$$\sum_{i=0}^{M-1} 2^{-l_i} \leq 1$$

Constrained Optimization

problem for minimizing
average code length.

$$\min \sum_{i=0}^{M-1} p_i l_i$$

$$\text{s.t.} \quad \sum_{i=0}^{M-1} 2^{-l_i} \leq 1$$

objective

constraint

↓
We can use Lagrange multiplier
framework to solve problem above

$$F = \sum_{i=0}^{M-1} P_i l_i + \lambda \left(\sum_{i=0}^{M-1} 2^{-l_i} - 1 \right)$$

Lagrange
multiplier

$$\frac{\partial F}{\partial P_i}$$

Differentiate
wrt to each P_i and
set equal to 0.

constraint

$$P_i + \lambda 2^{-l_i} (-1) \ln 2 = 0$$

$$\Rightarrow 2^{-l_i} = \frac{P_i}{\lambda \ln 2}$$

$$\Rightarrow l_i = \log_2 \frac{\lambda \ln 2}{P_i}$$

optimal codeword length.

Still have to find λ .

To find λ use constraint

$$\sum_{i=0}^{M-1} 2^{-l_i} = 1$$

substitute l_i from (#) $\Rightarrow$

$$\sum_{i=0}^{M-1} 2^{-\log_2 \frac{\lambda \ln 2}{p_i}} = 1$$

$$\Rightarrow \sum_{i=0}^{M-1} \frac{p_i}{\lambda \ln 2} = 1$$

$$\Rightarrow \boxed{\lambda \ln 2 = 1}$$

$$\Rightarrow \boxed{l_i^* = \log_2 \frac{1}{p_i}}$$

Optimal codeword length, which minimizes average code length

$$l_i^* = \log_2 \left(\frac{1}{P_i} \right)$$

inversely proportional to P_i

Larger probability
 $\Rightarrow$ lower codeword length.

Efficient Representation

Ex: Morse Code:

Telegraph System

one of the most popular source codes for English alphabet.

Dot
Dash
Binary code.

Length of codeword
~ inversely proportional to
natural frequency of
particular symbol.

